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SEMI-ANALYTICAL FINITE ELEMENT ANALYSIS OF END PROBLEMS FOR ORTHOTROPIC CYLINDERS

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Abstract—A semi-analytical finite element method has been developed for the solution of general asymmetric elastostatic end problems for orthotropic cylinders by an eigensolution technique. The finite element equations for the semi-infinite cylinder are derived from the potential energy expression assuming that the solution decays exponentially in the axial direction. The resulting quadratic matrix eigenvalue problem yields the decay parameters of the exponential solutions as eigenvalues. For finite cylinders two sets of eigenfunctions together with zero-eigenvalue solutions, if applicable, are superposed to satisfy the end conditions in a least square sense. Two cases of three dimensional flexure analysis of an orthotropic beam of hollow circular section have been considered for illustration.

Accurate results have been obtained using this technique which lends itself to easy programming for a digital computer. Although the method is less versatile than the three dimensional finite element methods, it may prove to be a powerful and economical method for a wide range of elasticity problems.

NOTATION

r_0	outer radius of the cylinder
r_i	inner radius of the cylinder
a	normalized inner radius of the cylinder
u, v, w	normalized displacement components
U, V, W	displacement Eigenfunctions
$[C]$	elasticity matrix
λ	eigenvalue
$2l$	normalized length of the cylinder
$[k_e], [k_{1e}], [k_{2e}], [k_{3e}]$	element stiffness matrices
$[K_1], [K_2], [K_3]$	global stiffness matrices
f_k^0	prescribed end value for the kth quantity
f_k^i	ith eigenfunction for the kth quantity

INTRODUCTION

The rigorous linear elastostatic solution of a cylinder under arbitrary loading may, in general, be obtained in two steps. First a solution satisfying the boundary conditions on the lateral surface is derived without considering the end conditions. Then a second solution is derived involving the arbitrary end conditions and homogeneous conditions on the lateral surface. This step is probably more difficult and has attracted considerable attention in recent years. The solution for a semi-infinite cylinder, which can be used to get the solution for a cylinder of finite length, under arbitrary end conditions can be obtained in terms of a set of eigenfunctions decaying exponentially along the axis of the cylinder and satisfying the homogeneous boundary conditions on the curved surfaces. This problem has been the subject of many recent investigations. The end problem for a cylinder has been extensively investigated for isotropic[1–3] and transversely isotropic[4, 5] materials by classical eigensolution techniques[6, 7]. The solutions are obtained in terms of Bessel functions. Byrnes[8] presented a Frobenius type series solution for the cylindrically orthotropic axisymmetric cylinder. Otherwise such problems are solved by fully discretized three dimensional finite element methods[9]. The former solution method showed slow convergence whereas the latter requires solution of a large number of equations. This motivates the development of a solution technique which would numerically find the individual eigenfunctions and yet retain the analytical nature of eigenfunction expansion technique.

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In this work the cylinder is discretized only in the radial direction using tubular elements while the dependence of the solution on the axial and circumferential coordinate directions is taken in terms of continuous analytical functions. The radial eigenfunctions are obtained as piece-wise low order polynomials by solving a quadratic matrix eigenvalue problem. The end problem is finally solved by a discrete least square technique. Recently Rukos[10] presented a similar formulation where he solved two dimensional problems and indicated a solution procedure for isotropic cylinders.

FORMULATION OF THE PROBLEM

The eigenvalue problem for the semi-infinite hollow orthotropic cylinder is formulated in terms of the three displacement components and the principle of minimum potential energy is used in conjunction with a piece-wise Rayleigh–Ritz method. Consider a semi-infinite cylinder of outer and inner radii r_0 and r_i. The cylindrical coordinate system is shown in Fig. 1. The material is cylindrically orthotropic with the $\hat{z}$-axis being the axis of anisotropy and a plane normal to the $\hat{z}$-axis being a plane of elastic symmetry. $\hat{u}(\hat{r}, \theta, \hat{z})$, $\hat{v}(\hat{r}, \theta, \hat{z})$ and $\hat{w}(\hat{r}, \theta, \hat{z})$ are the three displacement components. Non-dimensional coordinates (r, θ, z) are used and the following normalizations are carried out.

$$r = \hat{r}/r_0, \quad z = \hat{z}/r_0, \quad a = r_i/r_0$$
$$u = \hat{u}/r_0, \quad v = \hat{v}/r_0, \quad w = \hat{w}/r_0. \tag{1}$$

The constitutive relations are:

$$\{\epsilon\} = \mathrm{L}(u, v, w)$$
$$\{\sigma\} = [C]\{\epsilon\} \tag{2}$$

where

$$\{\sigma\}^T = \{\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \tau_{rz}, \tau_{\theta z}, \tau_{r\theta}\}$$
$$\{\epsilon\}^T = \{\epsilon_{rr}, \epsilon_{\theta\theta}, \epsilon_{zz}, \gamma_{rz}, \gamma_{\theta z}, \gamma_{r\theta}\}.$$

The detail expressions for the above linear strain-displacement relations and the stress-strain relations for cylindrically orthotropic materials are given in Ref. [11].

The displacement functions, in terms of harmonic functions in the circumferential direction, are taken as follows.

$$u(r, \theta, z) = \sum_m U_m(r) \exp(im\theta - \lambda_m z)$$
$$v(r, \theta, z) = -i \sum_m V_m(r) \exp(im\theta - \lambda_m z) \tag{3}$$
$$w(r, \theta, z) = \sum_m W_m(r) \exp(im\theta - \lambda_m z)$$
$$|m| = 0, 1, 2, \ldots.$$

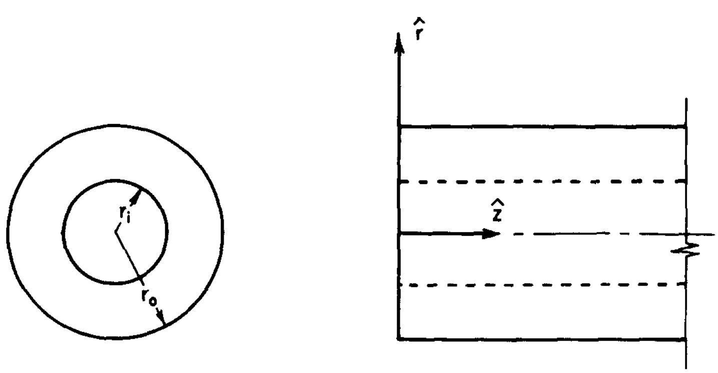

Fig. 1. The semi-infinite cylinder.

The underlying assumption is that the prescribed end conditions can be expanded in Fourier series in the circumferential coordinate. This uncouples the harmonics and each term can be treated separately. Henceforth the subscript m is dropped. The task is to find a set of eigenvalues λ and corresponding eigenfunctions $U(r)$, $V(r)$ and $W(r)$ that satisfy the equilibrium equations and the homogeneous boundary conditions on the curved surfaces at $r = a$ and $r = 1$.

The homogeneous boundary conditions at $r = a$ and $r = 1$ are:

$$\begin{aligned} &u \text{ or } \sigma_{rr} = 0 \\ &v \text{ or } \tau_{r\theta} = 0 \\ &w \text{ or } \tau_{rz} = 0. \end{aligned} \tag{4}$$

For a solid cylinder (i.e. $a = 0$) the conditions at the center are:

$$\begin{aligned} u = v = \tau_{rz} = 0 \quad &\text{for} \quad m = 0 \\ \sigma_{rr} = \tau_{r\theta} = w = 0 \quad &\text{for} \quad m = 1 \\ u = v = w = 0 \quad &\text{for} \quad m > 1. \end{aligned} \tag{5}$$

The total potential energy is:

$$\Pi = \frac{1}{2}\iiint_{\text{Vol}} \{\sigma\}^T\{\epsilon\}\,\mathrm{d}v - \iint_{AE} (\tau^0_{rz}u + \tau^0_{\theta z}v + \sigma^0_{zz}w)\,\mathrm{d}A \tag{6}$$

where τ^0_{rz}, $\tau^0_{\theta z}$ and σ^0_{zz} denote boundary stresses and AE refers to the two ends, $z = 0$ and $z = \infty$.

From the principle of minimum potential energy

$$\delta\Pi = 0. \tag{7}$$

In the present work the radial eigenfunctions $U(r)$, $V(r)$ and $W(r)$ are approximately determined as piece-wise cubic polynomials through the following finite element discretization.

FINITE ELEMENT DISCRETIZATION

The cylinder is subdivided into tubular elements over which the radial functions are approximated as cubic polynomials. The variation of displacement components in the circumferential and axial directions are according to eqn (3). Then the problem reduces to a one-dimensional finite element analysis. The nodal variables of the element (Fig. 2) are the

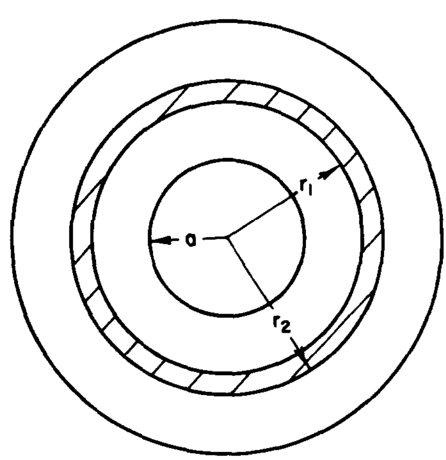

Fig. 2. The finite element.

functions and their r-derivatives. Inside the 12 degrees of freedom element the displacement eigenfunctions are expressed as:

$$\begin{Bmatrix} U(r) \\ V(r) \\ W(r) \end{Bmatrix} = \begin{bmatrix} U(r_1) & U'(r_1) & U(r_2) & U'(r_2) \\ V(r_1) & V'(r_2) & V(r_2) & V'(r_2) \\ W(r_1) & W'(r_1) & W(r_2) & W'(r_2) \end{bmatrix} \begin{Bmatrix} \phi_1(r) \\ \phi_2(r) \\ \phi_3(r) \\ \phi_4(r) \end{Bmatrix} . \tag{8}$$

Here primes, $(\ )'$, denote differentiation with respect to r and ϕ's are the cubic Hermite interpolating polynomials.

ELEMENT STIFFNESS MATRIX

The potential energy of the element can be written as:

$$\Pi_e = \frac{1}{2}\iiint_{V_e} \{\sigma\}^T\{\epsilon\}\,\mathrm{d}v - \iint_{A_e} (\tau_{rz}^0 u + \tau_{\theta z}^0\, v + \sigma_{zz}^0 w)\,\mathrm{d}A \tag{9}$$

where V_e refers to the volume of the element and A_e to its two ends.

Using eqn (2), Π_e can be expressed as a functional of the displacement components. By varying Π_e and integrating by parts to eliminate the derivatives of u, v and w with respect to θ and z, we finally get for a particular mth harmonic:

$$\begin{aligned}
\delta\Pi_e = \frac{\pi}{2\lambda}\Bigg[\int_{r_1}^{r_2} &\{c_{11}r(U'\delta U') + c_{12}(U\delta U' + U'\delta U) + mc_{12}(U'\delta V + V\delta U') + (c_{22} + m^2 c_{66})r^{-1}(U\delta U) \\
&+ m(c_{22} + c_{66})r^{-1}(U\delta V + V\delta U) + (m^2 c_{22} + c_{66})r^{-1}(V\delta V) + c_{44}r(W'\delta W') + m^2 c_{55}r^{-1}(W\delta W) \\
&+ c_{66}r(V'\delta V') - mc_{66}(U\delta V' + V'\delta U) - c_{66}(V\delta V' + V'\delta V)\}\,\mathrm{d}r \\
&+ \lambda\int_{r_1}^{r_2} \{c_{13}r(U'\delta W - W\delta U') + c_{23}(U\delta W - W\delta U) - m(c_{23} + c_{55})(W\delta V - V\delta W) \\
&\qquad + c_{44}r(W'\delta U - U\delta W')\}\,\mathrm{d}r \\
&+ \lambda^2\int_{r_1}^{r_2} \{-c_{33}r(W\delta W) - c_{44}r(U\delta U) - c_{55}r(V\delta V)\}\,\mathrm{d}r\Bigg] + \Bigg[\iint_{A_e} \{[\tau_{rz} - \tau_{rz}^0)\delta U \\
&+ (\tau_{\theta z} - \tau_{\theta z}^0)\delta V + (\sigma_{zz} - \sigma_{zz}^0)\delta W\}\,\mathrm{d}A\Bigg].
\end{aligned} \tag{10}$$

The expression for $\delta\Pi_e$ is shown here for the real part of θ function corresponding to the mth harmonic. The other part will come out similarly.

The significance of carrying out the integration with respect to z, in the context of Rayleigh–Ritz method, should be noted here. The boundary conditions at the z-boundaries (i.e. $z = 0$ and $z = \infty$) are arbitrary at this stage. Hence the integrands in the surface integrals (the bracketed terms at the end of eqn (10)) can be identically set to zero. This implies that the prescribed conditions on the z-surfaces are such as to satisfy the boundary integrals identically. This is a distinguishing feature of the eigenvalue formulation as against the formulation of a boundary value problem where all of the prescribed boundary conditions are known beforehand.

At the interelement boundary the displacements and their r-derivatives, which enter into the potential energy expression, are matched. With the assumed displacement shape in eqn (8), the element stiffness matrix can be computed from eqn (10) by numerical integration. The element stiffness matrix comes out in the following form.

$$[k_e] = [k_{1e}] + \lambda[k_{2e}] + \lambda^2[k_{3e}], \tag{11}$$

where $[k_{1e}]$ and $[k_{3e}]$ are (12×12) real symmetric matrices while $[k_{2e}]$ is skew-symmetric, which is obvious from eqn (10).

THE EIGENVALUE PROBLEM

Assembling the element stiffness matrices, and imposing the homogeneous displacement boundary conditions, if any, we arrive at the global stiffness matrix. Since the discretization is only in one direction the assembly is straightforward. Thus the finite element formulation leads to a quadratic matrix eigenvalue problem:

$$[[K_1]+\lambda[K_2]+\lambda^2[K_3]]\{y\}=0 \tag{12}$$

where $\{y\}$ is the vector of nodal variables. The matrices are narrow banded and again $[K_1]$ and $[K_3]$ are real symmetric while $[K_2]$ is skew-symmetric. Equation (12) in general possesses complex solutions along with the corresponding complex eigenvalues. It may be noted that the eigenvalues λ's occur in conjugate pairs and so do the eigenfunctions. The stresses corresponding to the eigenfunctions are self-equilibrated.

It may be noted that $\lambda = 0$ leads to a non-trivial solution of eqn (12) for certain combinations of boundary conditions on the curved surfaces. They correspond to the uniform stress states and the rigid body displacement modes. These problems can be solved analytically without resorting to finite element discretization. They represent the generalized plane strain solutions[11].

An alternate to the foregoing Rayleigh–Ritz technique would be to consider the three partial differential equations in terms of displacements (u, v, w) and reduce them to one dimensional equations using the solution of form eqn (3) and using a piece-wise Galerkin procedure. While applying the Galerkin technique the boundary residuals corresponding to the ends should be identically set to zero.

In the analytical solution of end problems[3, 12] the completeness of the eigenfunctions have been proved for only certain combinations of boundary conditions on the curved surfaces and the orthogonality relation among the eigenfunctions have been developed for different cases. However no such work exists for the semi-analytical formulation developed in the present work.

SOLUTION OF A BOUNDARY VALUE PROBLEM

The solution to the cylinder with given end conditions is sought as an expansion of the eigenfunctions discussed above. In this work a least square criterion is used to satisfy the prescribed end conditions. The solution for a finite cylinder is indicated here. The semi-infinite cylinder forms a special case of this. Consider the cylinder of normalized length $2l$ (Fig. 3) subjected to prescribed tractions (τ_{rz}, $\tau_{\theta z}$, σ_{zz}) and/or displacements (u, v, w) $f_1^0, f_2^0, \ldots, f_6^0$ at the two ends.† It is assumed that the prescribed quantities can be expanded in Fourier series in the circumferential coordinate. This uncouples the harmonics and each term of the Fourier series

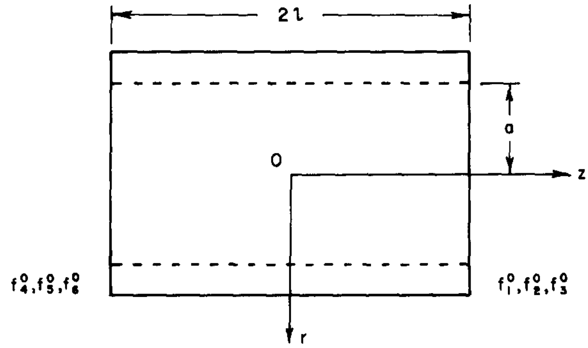

Fig. 3. The finite cylinder.

†f_1^0 and f_4^0 are $u(r, \theta)$ and/or $\tau_{rz}(r, \theta)$, etc.

can be solved separately. So the solution corresponding to one single harmonic will be considered.

The finite cylinder, in Fig. 3, can be considered as the intersection of two semi-infinite regions. Considering N eigenfunctions the stress or displacement, $X(r, z)$, at any point may be expressed as:

$$X(r, z) = \sum_{i=1}^{N} X^i(r) \exp(-\lambda_i l)\{A_i \exp(-\lambda_i z) + pB_i \exp(\lambda_i z)\} + \sum_{j=1}^{N_0} C_j X_0^j(r, z) \tag{13}$$

where $X^i(r)$ is the ith eigenfunction. The displacement eigenfunctions are converted to stress eigenfunctions using eqn (2). A_i and B_i are eigenfunction expansion coefficients. p is +1 or −1 depending upon the particular quantity represented by $X(r, z)$. $X_0^j(r, z)$ represents zero-eigenvalue solutions which includes the elementary generalized plane strain solutions and rigid body displacement modes. N_0 is the number of zero-eigenvalue solutions considered. The $\exp(-\lambda_i l)$ term is used for normalization.

The integral of the square of the boundary residuals at the two ends, $E(A, B, C)$, can be written as:

$$E(A, B, C) = \sum_{k=1}^{3} \left[\int_a^1 \left\{f_k^0 - \sum_{i=1}^{N} f_k^i(A_i \exp(-2\lambda_i l) + p_k B_i) - \sum_{j=1}^{N_0} C_j f_{0k}^j(r, l)\right\}^2 r\,\mathrm{d}r\right]$$
$$+ \sum_{k=4}^{6} \left[\int_a^1 \left\{f_k^0 - \sum_{i=1}^{N} f_k^i(p_k A_i + B_i \exp(-2\lambda_i l)) - \sum_{j=1}^{N_0} C_j f_{0k}^j(r, -l)\right\}^2 r\,\mathrm{d}r\right] \tag{14}$$

where f_k^i is the ith eigenfunction corresponding to the quantity f_k^0.

Minimizing E with respect to the expansion coefficients (A_i, B_i, C_j), we arrive at a set of linear equations:

$$[D]\{\alpha\} = \{F\} \tag{15}$$

where $[D]$ is a symmetric positive definite matrix, $\{\alpha\}$ consists of the expansion coefficients and the prescribed end conditions are used to find the elements of $\{F\}$.

For a general problem where the end stresses are not self-equilibrated the applicable non-decaying elementary solution, which is a zero-eigenvalue solution, has to be considered. In a prescribed displacement problem it is generally necessary to consider appropriate rigid body modes along with the other eigenfunctions.

COMPUTATIONAL ASPECTS

The computer implementation of the solution technique does not pose any specific problem. The integrals constituting the elements of the element stiffness matrix are identified symbolically and a fourth order Gauss quadrature formula is used for integration. It gives exact results for the polynomial integrals involved. The quadratic matrix eigenvalue problem in eqn (12) is solved by an iterative technique used in Ref. [5]. The global stiffness matrix in eqn (12) is block tridiagonal where the block dimension is six. The determinant value is computed by block tridiagonal reduction.

The approximate position of the eigenvalues on the λ-plane are found from the "Determinant Maps". Here the computer printer is used to indicate at each grid point in a region of the λ-plane, the quadrant location of the determinant value. As the eigenvalues occur in conjugate pairs only the first quadrant of the λ-plane is searched. The approximate position of a complex root lies on the location where all four quadrants meet on the map[13]. The real root is indicated by a changeover from the first to the second quadrant. The approximate roots are then refined by a Newton–Raphson iterative technique. The procedure starts with two trial points and the third point is taken as the average of the two. A quadratic function is fitted through them to predict the root. Then successively the three best solutions are used in each iteration till convergence is achieved. The use of determinant maps ensures the determination of all roots within the area searched[13]. For most of the cases the pattern of the eigenspectrum can be guessed after locating the first few roots. Once an eigenvalue is determined, the eigenvectors are computed by back substitution.

The least square method for minimizing the boundary residuals can be programmed in a compact form. Here again a fourth order Gauss quadrature formula has been used within each element to compute the inner products. As the eigenvalues occur in conjugate pairs, each complex eigenfunction is split into real and imaginary parts and the whole computation is carried out in the real mode. In general an orthotropic cylinder has a mixture of real and complex eigenvalues and hence carrying out the computation in complex mode is not advantageous.

NUMERICAL EXAMPLES

The semi-analytical formulation presented here yields accurate results. The eigenvalues for solid cylinders of isotropic and transversely isotropic (magnesium) materials computed by this method are compared with the exact values in Table 1.

As the prescribed displacement end problem presents a severe test for the convergence of least square fitting, the three dimensional flexure analysis of hollow beams under prescribed end displacements is chosen for investigation. The generalized plane strain solution for orthotropic beams of hollow circular section are given by Lekhnitskii[11] which are used here as the zero-eigenvalue solutions. It may be noted that such solutions for a fully orthotropic cylinder predict a non-linear distribution of bending stresses through the thickness. However for the particular material constants used here the distribution is practically linear.

Table 1. Comparison of eigenvalues determined by F.E.M. with the exact results for solid isotropic and transversely isotropic cylinders

	Magnesium (T.I) for m = 0	
	F.E.M with 10 elements	Exact [4]
λ_1	2.8768 + i 1.0817	2.8768 + i 1.0817
λ_2	4.8911	4.8911
λ_3	7.3723 + i 0.9543	7.3723 + i 0.9543
λ_7	16.1799 + i 0.9505	16.1793 + i 0.9486
λ_{10}	22.8500	23.0069
	Isotropic (ν = 0.25) for m = 2	
	F.E.M with 20 elements	Exact [1]
λ_1	2.10435 + i 0.959221	2.10435 + i 0.959221
λ_5	9.03528 + i 1.82446	9.03528 + i 1.82446
λ_{10}	17.8429	17.8428
λ_{15}	24.9836 + i 2.30646	24.9818 + i 2.30918
λ_{20}	33.6613	33.6564

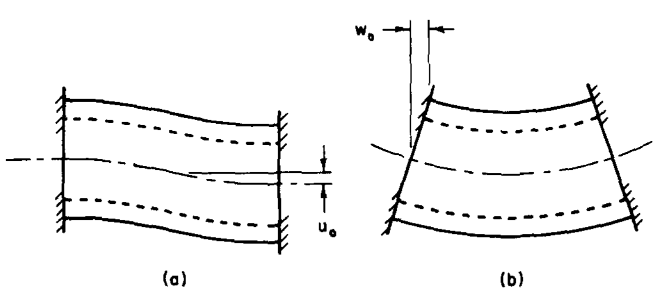

Fig. 4. The example problems.

As a first example consider the three dimensional analysis of a beam undergoing prescribed end displacements (Fig. 4a). The end conditions are:

$$u = u_0 \cos\theta, \quad v = -u_0 \sin\theta, \quad w = 0. \tag{16}$$

The cylindrically orthotropic elastic constants (for Topaz), used in Ref. [14] are used here. They are as follows:

$$\begin{aligned} c_{11} &= 2.8650085E, & c_{22} &= 3.5582351E, & c_{44} &= 1.3456752E \\ c_{12} &= 1.284508E, & c_{23} &= 0.89711678E, & c_{55} &= 1.1010070E \\ c_{13} &= 0.85633875E, & c_{33} &= 2.9971856E, & c_{66} &= 1.3358322E \end{aligned} \tag{17}$$

where $E = 10^6\ \mathrm{kg/cm^2}$.

The non-dimensional inside radius a of the cylinder is taken as 0.80. The problem is solved for two cases, namely $l = 1$ and $l = 0.5$. The curved surfaces are free of stresses. Thus the boundary conditions at $r = a$ and $r = 1$ are:

$$\sigma_{rr} = \tau_{r\theta} = \tau_{rz} = 0. \tag{18}$$

The eigenvalues corresponding to the flexural mode (i.e. $m = 1$) are computed with thirty-two elements for the above mentioned problem. To improve the accuracy of the stress boundary condition the element size at the two ends has been taken as half the size of the otherwise uniform grid. The first thirty eigenvalues, in the order of increasing real parts, are listed in Table 2.

An appropriate zero-eigenvalue solution for the problem is a constant shear solution and it is taken from Section 53 of Ref. [11]. The following rigid body displacement mode has also been considered in the least square formulation

$$u = z\cos\theta, \quad v = -z\sin\theta, \quad w = -r\cos\theta. \tag{19}$$

The least square fitting of the end displacements have been done with thirty eigenfunctions besides the zero-eigenvalue solutions. Since the problem has one axis of symmetry, the size of

Table 2. Eigenvalues for the hollow orthotropic cylinder for the $m = 1$ case

3.4458928 + i 2.8398036	121.14588
17.511591	134.52633 + i 60.791867
19.630588 + i 12.974288	138.44460
34.190662 + i 18.956807	148.86214 + i 66.771338
34.709759	155.74450
48.531635 + i 24.916544	163.19921 + i 72.750770
51.977071	173.04554
62.860247 + i 30.893804	177.53795 + i 78.730154
69.261746	190.34788
77.191406 + i 36.873498	191.87891 + i 84.709448
86.553402	206.22278 + i 90.688586
91.523981 + i 42.852303	207.65184
103.84859	220.57037 + i 96.667460
105.85738 + i 48.832812	224.95789
120.19147 + i 54.812361	234.92270 + i 102.64590

$[D]$ matrix in eqn (15) is halved and a (49×49) real symmetric matrix is solved to get the results. The least square fit of prescribed end quantities at the end $z = l$ are shown in Table 3. For the purpose of graphical representation the stresses are normalized with respect to σ_0, where σ_0 is the maximum bending stress from the elementary theory. The normalized stresses are denoted by a bar. The variation of extreme fiber stress σ_{zz} along the span for the case with $l = 1.0$ is shown in Fig. 5. The distribution of σ_{zz}, $\sigma_{\theta\theta}$, τ_{rz}, $\tau_{\theta z}$ through the thickness for the same case are shown in Figs. 6 and 7. The values from elementary theory are also plotted alongside for comparison. For the case with $l = 0.5$ only the σ_{zz} variation along the span is shown in Fig. 8.

Table 3. Least square fit of prescribed end displacements in the first example for the case with $l = 1.0$

r	u/u_0	v/u_0	w/u_0	% error in u	% error in v
0.80	0.99984	-0.99927	7.28×10^{-5}	-0.0162	0.0727
0.84	0.99996	-1.00006	-2.62×10^{-5}	-0.0044	-0.0060
0.88	1.00000	-0.99995	8.90×10^{-5}	0.0001	0.0058
0.92	1.00005	-1.00005	-2.32×10^{-5}	0.0050	-0.0050
0.96	0.99999	-0.99994	-1.74×10^{-4}	-0.0009	0.0057
1.00	0.99785	-1.00067	-1.36×10^{-3}	-0.2153	-0.0674

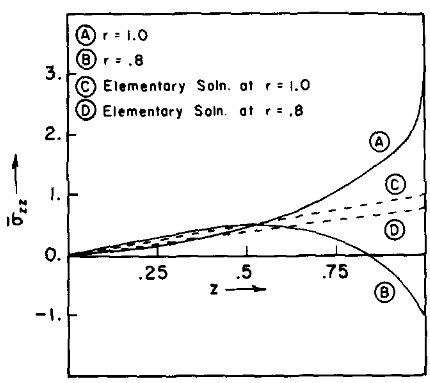

Fig. 5. Variation of σ_{zz} along the span in the first example for the case with $l = 1.0$.

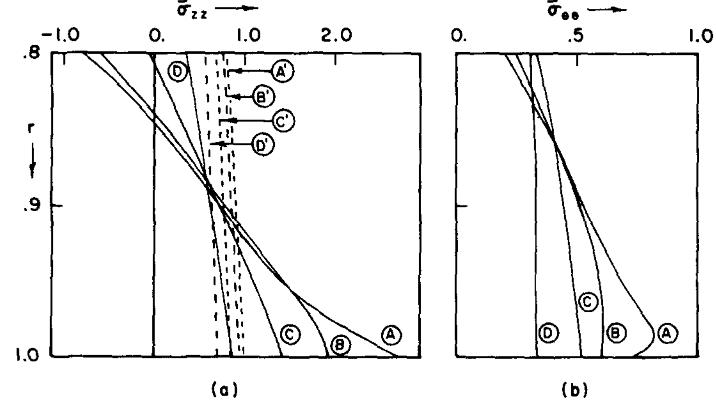

Fig. 6. Distribution of normal stresses through the thickness at various sections in the first example for the case with $l = 1.0$. A: $z = 0.99$, B: $z = 0.95$, C: $z = 0.85$, D: $z = 0.70$; A', etc. are the corresponding distributions for elementary solution[11].

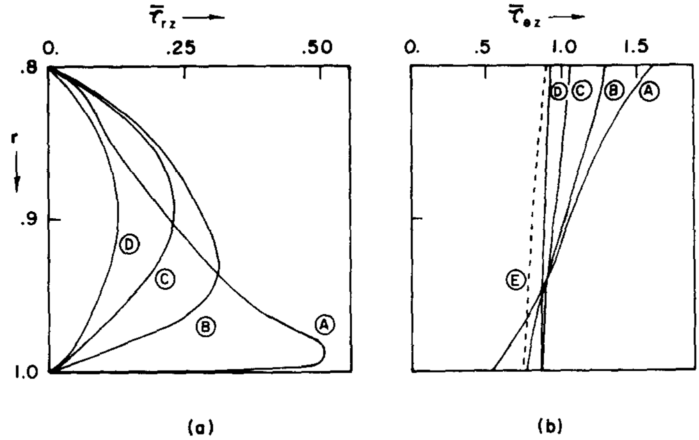

Fig. 7. Distribution of shear stresses through the thickness at various sections in the first example for the case with $l = 1.0$. A: $z = 0.99$, B: $z = 0.95$, C: $z = 0.85$, D: $z = 0.70$, E: elementary solution.

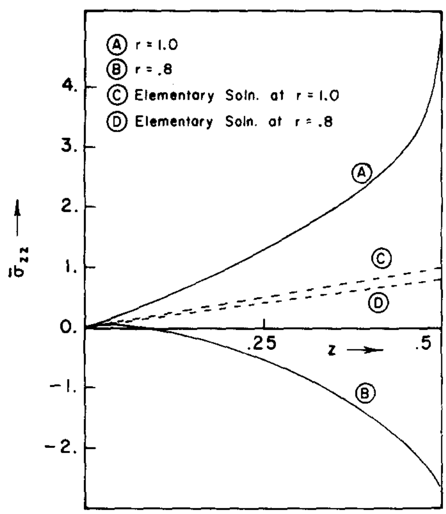

Fig. 8. Variation of σ_{zz} along the span in the first example for the case with $l = 0.5$.

As a second example consider the three dimensional analysis of a hollow orthotropic beam undergoing prescribed end rotation (Fig. 4b). The end conditions are:

$$u = v = 0, \quad w = w_0 r \cos\theta. \tag{20}$$

The material constants, section geometry and the conditions on the curved surfaces are the same as those used in the first example. So the eigenvalues and eigenfunctions are identical. The zero-eigenvalue solution corresponding to the uniform moment problem is derived in Section 41 of Ref. [11]. The following rigid body mode is considered in the least square formulation along with thirty eigenfunctions.

$$u = \cos\theta, \quad v = -\sin\theta, \quad w = 0. \tag{21}$$

As before, because of symmetry, the size of $[D]$ matrix in eqn (15) is halved. The accuracy of the solution obtained is comparable to that achieved in the first example. The along-the-span variation of extreme fiber bending stress σ_{zz} for two cases, namely $l = 1.0$ and $l = 0.5$, are shown in Figs. 9 and 10. The detailed results for the above examples are in Ref. [15].

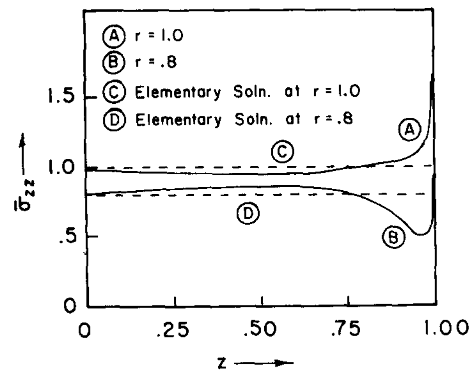

Fig. 9.

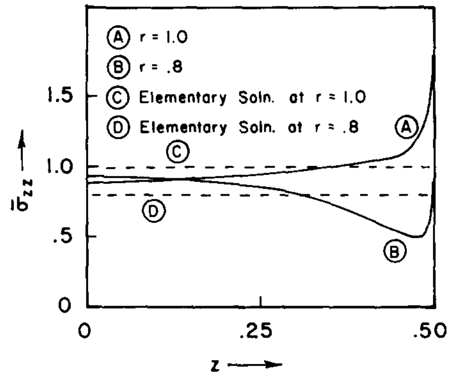

Fig. 10.

Fig. 9. Variation of σ_{zz} along the span in the second example for the case with $l = 1.0$.

Fig. 10. Variation of σ_{zz} along the span in the second example for the case with $l = 0.5$.

DISCUSSION

The accuracy of the eigenvalues obtained through the finite element method presented here is evident from the result presented in Table 1. The stress boundary conditions are very accurately satisfied on the curved surfaces. In the example problems, where thirty eigenfunctions are used, the orders of magnitudes of the errors in the final solutions are three to four times lower than that of the maximum interior stresses. It may be mentioned that much less accurate results are obtained, particularly in the case of higher modes, when linear shape functions are used instead of cubic polynomials[15].

The prescribed displacement end problem considered in the last section, as stated earlier, presents a severe test for the convergence of eigenfunction series. The accuracy of the results obtained is illustrated in Table 3. Though the material used in the examples is not highly anisotropic, the extent of the end zone particularly in the first example is significant. There, in the case with $l = 0.5$, the end effect is evident over the entire span. The stresses deviate considerably from the elementary solution. Also over a considerable portion of the span in the first example, the bending stresses over a part of the cross section have an opposite sign compared to the results predicted by the elementary theory.

Due to rigid bonding of the ends the stresses are singular around the periphery ($r = a$ and $r = 1$) at the two ends. This has not been explicitly considered here. However the trend of the fitted stress profile (Figs. 5, 8–10) indicates this singular behavior. Hence the stresses very near the singular points found from this analysis will not be accurate. Also since the displacements are fitted at the end sections, in an approximate solution formulated in terms of displacement eigenfunctions, the stresses very near the ends do not converge properly especially due to the presence of higher modes.

In the first example the net shear force is increased by a factor of 2.1365 to produce same lateral deflection, when the span $2l$ is decreased from 2.0 to 1.0. Similarly in the second example the net effective bending moment required to produce certain end rotation increased by a factor of 2.0308 when the span is halved from 2.0 to 1.0. Both the values corresponding to the elementary solutions are 2.0. This non-linear dependence of net force on the aspect ratio was also observed in Ref. [5] in connection with the axisymmetric stress analysis of transversely isotropic cylinders. The example problems chosen can be considered as short, thick orthotropic cylindrical shells. This technique presents an attractive method for a three dimensional solution of such problems.

A similar end problem with force boundary conditions reported in Ref. [9] required the solution of a banded matrix equation involving about 800 unknowns. This implies that the final solution for determination of expansion coefficients by the present method would require about fifteen times less computational effort. The method, however, calls for prior generation of eigenfunctions, but they can again be used to solve other end problems with identical section geometry and material property.

The basic idea behind the present method is to get a product-type solution wherein an exact functional form is assumed in one or more coordinate directions and the functional form along the remaining coordinates is approximated by a discrete numerical formulation. This idea has been used with varied terminologies such as the semi-analytical finite element process[16] or separation of variable method[17] for linear boundary value problems where assumed continuous functions satisfy the relevant boundary conditions. The present formulation in terms of eigenfunctions permit arbitrary boundary conditions at the end of the cylinder and hence offers an effective method for the rigorous solution of a class of three dimensional problems. Such a method would be more economical than general purpose three dimensional finite element methods. Rukos[10] presented a semi-analytical method, termed continuous finite element, which is very similar to the present technique. He obtained a set of ordinary differential equations in one coordinate by using finite element discretization in the other direction and applying a piece-wise Galerkin technique. He then solved them in terms of exponential functions. This is mathematically equivalent to the method presented here using a potential energy formulation and Rayleigh–Ritz technique. Although Rukos indicated the formulation for three dimensional isotropic elasticity problems he solved only two dimensional Laplace and elasticity problems. The method used by him to fit the end conditions is essentially a collocation method where element nodes are station points. Such a method is less versatile than the least square technique used in the present work. Moreover he did not discuss the role of zero-eigenvalue solutions in general problems.

This technique has been applied to the plane strain and plane stress analysis of inhomogeneous anisotropic strips[15] and can be applied to other geometries such as cones and spherical shell segments. Finally this type of solution can also be constructed for other types of bodies such as rectangular prisms. There, of course, the finite element analysis will involve two dimensions and the efficiency of such a method is yet to be determined.

CONCLUSIONS

The semi-analytical finite element method for the linear elastostatic analysis of orthotropic cylinders, presented here, is an effective method for the solution of a class of elasticity problems. The solution technique lends itself to easy programming on a digital computer. The finite element modelling using cubic polynomials yields accurate eigensolutions and the least square procedure has been found to be effective in the evaluation of the coefficient of an eigenfunction expansion.

Although the technique is less versatile than the three dimensional finite element methods, it may prove to be a powerful and economical method for a wide range of elasticity problems. The proposed method permits material property variation along the direction of finite element discretization. For any given problem, the eigenvalue analysis need be carried out only once and only the final solution which entails the evaluation of the coefficients of eigenfunction expansion using the least square procedure, need be repeated for each set of end conditions.

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